THE CHAOTIC PROPERTIES OF $Q-$STATE POTTS MODEL ON THE BETHE LATTICE: $Q < 2$

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The $Q-$ state Potts model on the Bethe lattice is investigated for $Q < 2$. The magnetization of this model exhibits a complicated behavior including both the period doubling bifurcation and chaos. The Lyapunov exponents of the Potts-Bethe map are considered as order parameters. We find a scaling behavior in the distribution of Lyapunov exponents in fully developed chaotic case. Using the thermodynamic formalism of dynamical systems have we investigated the nonanalytic behavior in the distribution of Lyapunov exponents and located the point of phase transition of the "chaotic free energy".

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1. INTRODUCTION

The Q-state Potts model is one of generalizations of the Ising model constructed for investigation of the phase transition [1]. This model was initially defined for an integer Q but it has also many applications for noninteger Q. The exact solution of the two-dimensional Potts model for general Q has been obtained only at the self-dual point by mapping it into two-dimensional inhomogeneous six vertex model [2].

Another exact solution of the Potts model for general Q and coordination number can be obtained on the Bethe lattice. It is interesting to note that similar results can be obtained in the $N \to 1$ limit of $N \times N$ Hermitian matrix model on random graphs [3].

The properties of the ferromagnetic and antiferromagnetic Potts models in the magnetic field have been rigorously considered on the Bethe lattice by means of a recursion relation [4].

The Q-state Potts model on the Bethe lattice has been recently investigated for $Q < 2$ [5]. Many physical processes can be formulated in terms of Q-state Potts model when $Q < 2$, e.g. the resistor network, dilute spin glass, percolation and Self Organizing Critical systems [1,7,6]It has been shown that when $Q < 2$ the Q-state Potts model exhibits a large verity of phase transitions leading to specially modulated and chaotic phases. These phases are similar to those obtained in Axial Next Nearest Ising, chiral Potts and three-site interacting Ising models. It is necessary to point out that contrary to the above mentioned models these phase in the Potts model ($Q < 2$) are obtained without the frustration [8–10].

In this paper we investigate the Q-state Potts model on the Bethe lattice for $Q < 2$ using the thermodynamic formalism of multifractals.. The reason of phase transitions taking place in this model is that the attractor of the map (Potts-Bethe) which is used for calculation of average quantities has a complicated dependence on the parameters of the Potts model. Being one dimensional the Potts-Bethe map exhibits period doubling cascade, chaos, etc. By using thermodynamic formalism of multifractals one can calculate the distribution of local Lyapunov exponents of Potts-Bethe map in fully developed chaotic

case. The Lyapunov exponent is not only a good order parameter for transition to chaos, but also completely characterizes the system in chaotic states. The general aim of this paper is to find a scaling in the distribution of local Lyapunov exponents of the Potts-Bethe map. The multifractal approach allow one to map the problem of computation of Lyapunov exponent onto thermodynamics of one-dimensional spin model and interpret scaling in the distribution of Lyapunov exponent as a phase transition in an one-dimensional spin model [12–15]. We obtain the phase transition temperature by means of numerical calculations of the free energy of this model.

This paper is organized as follows. The Potts model on the Bethe lattice and its recursion relation is presented in Sec. 2. In Sec. 3 we discuss the phase structure of the Potts model on the Bethe lattice. In Sec. 4 we investigate the Potts-Bethe map for the case of fully developed chaos. By using the thermodynamic formalism of multifractals, the distribution of local Lyapunov exponent is obtained and the phase transition is analyzed in terms of the one-dimensional spin model. Finally, in Sec. 5 we summarize our results and comment on their implications for the study of other systems.

2. THE POTTS MODEL ON THE BETHE LATTICE AND THE RECURSION RELATION

The Potts model in the magnetic field is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{<i,J>} \delta(\sigma_i, \sigma_j) - H \sum_i \delta(\sigma_i, 1), \tag{1}$$

where σ_i takes the values 1,2,..., Q, the first sum goes over all edges and the second one over all sites on the lattice. Additionally, we use the notation $K = J/kT$, $h = H/kT$.

The partition function and single site magnetization is given by

$$\mathcal{Z} = \sum_{\{\sigma\}} e^{-\mathcal{H}/kT},$$

$$M =< \delta(\sigma_0, 1) >= \mathcal{Z}^{-1} \sum_{\{\sigma\}} \delta(\sigma_0, 1) e^{-\mathcal{H}/kT}, \tag{2}$$

where the summation goes over all configurations of the system.

When the Bethe lattice is cut apart at the central point, it is separated into γ identical branches. The partition function can be written as follows

$$\mathcal{Z}_n = \sum_{\{\sigma_0\}} exp\left\{h\delta(\sigma_0, 1)\right\}\left[g_n(\sigma_0)\right]^{\gamma}, \tag{3}$$

where σ_0 is the central spin and $g_n(\sigma_0)$ is the contribution of each lattice branch. The latter is expressed through $g_{n-1}(\sigma_1)$, i.e. the contribution of the same branch containing $n-1$ generations and starting from the site belonging to the first generation,

$$g_n(\sigma_0) = \sum_{\{\sigma_1\}} exp\left\{K\delta(\sigma_0, \sigma_1) - h\delta(\sigma_1, 1)\right\}\left[g_{n-1}(\sigma_1)\right]^{\gamma-1}. \tag{4}$$

Introducing the notation

$$x_n = \frac{g_n(\sigma \neq 1)}{g_n(\sigma = 1)}, \tag{5}$$

one can obtain the Potts-Bethe map

$$x_n = f(x_{n-1}, K, h), \qquad f(x, K, h) = \frac{e^h + (e^K + Q - 2)x^{\gamma-1}}{e^{K+h} + (Q-1)x^{\gamma-1}}. \tag{6}$$

The magnetization of the central site for the Bethe lattice with n generation can be written as

$$M_n =< \delta(\sigma_0, 1) >= \frac{e^h}{e^h + (Q-1)x_n^{\gamma}}. \tag{7}$$

3.THE PHASE STRUCTURE OF THE POTTS MODEL

Let us consider the magnetization of the central site. In order to achieve the thermodynamic limit we tend the number of generations to infinity ($n \to \infty$). The recursion relation (6) converge to a fixed point at every values of parameters h, K in ferromagnetic case ($K > 0$) and has only one period doubling in the antiferromagnetic case ($K < 0$) corresponding to a rise of antiferromagnetic order in different sublattices for $Q \geq 2$ [4]. The situation change drastically for $Q < 2$ in contrast to above case. For systems with Q values in the range

$Q < 2$ and with antiferromagnetic interactions or for systems with Q values in the range $Q < 1$ and with ferromagnetic interactions one obtains for M versus h bifurcation diagrams with the full range of period doubling cascade, chaos, etc. [5]. The Figure 1 shows plots of M versus h for anti-ferromagnetic case($K = -0.5$,$Q = 0.8$, $\gamma = 3$).

The Potts model has many specially modulated and chaotic phases when $Q < 2$. the presence of phase transitions is in obvious contradiction to the universality hypothesis. The transition to chaos is provided by Feigenbaum exponents which is well known to be an one-dimensional map (Fig. 1). It is interesting to note that the similar transition has been found in Ising model with three-site interaction [10,16]. The Potts model($Q < 2$) and three-site interacting Ising model have the same universal Feigenbaum exponents.

As was already mentioned the Lyapunov exponents are not only good order parameters for the transition to chaos but also completely characterize the system in chaotic states. In the next section we compute the distribution of Lyapunov exponents by using thermodynamic formalism of multifractals. The recursion relation of the critical phenomena of the Potts model and dynamical systems are similar. The thermodynamical formalism of multifractals connects the thermodynamical quantities of an one-dimensional spin model and dynamical properties of strange attractors [11–15].

4.POTTS-BETHE MAP IN THE CASE OF FULLY DEVELOPED CHAOS

In this section we impose two restrictions on the parameters in order to apply the thermodynamic formalism of multifractals to the Potts-Bethe map. The first one is regards only an odd coordination number γ for getting even function of x of the Potts-Bethe map. The second one is the requirement that

$$f(0) = -f(f(0)), \tag{8}$$

from which we obtain the following restrictions on h and K

$$\exp(h) = \frac{1 - \exp(2K) + 2\exp(K) - \exp(K)Q - Q}{2\exp(\gamma K)}. \tag{9}$$

The second one is needed for exhibiting the fully developed chaotic behavior of the Potts-Bethe map in the (h,K) plane.

For a crisis map (Eqs.(6), (9)) we want to describe the scaling properties of an attracting set for the sequences x_n which is in this case the interval I: $[-\exp(-h), \exp(-h)]$ (Fig.2). For an index n, I is partitioned into 2^n intervals or n-cylinders, these being the segments with identical symbolic-dynamics sequences of length n taken with respect to the maximum point (we follow here Ref. [13]). The inverse function of Eq.(6), $h = f^{-1}$, has two branches, h_{-1} and h_1 as shown in (Fig.2) and the n-cylinders are all the nth-order preimages of I. The length of the cylinders is denoted by $l_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} \equiv h_{\epsilon_1} \circ h_{\epsilon_2} \circ \ldots \circ h_{\epsilon_n}(I)$ where $\epsilon \in \{-1, 1\}$.

Let us consider an one-dimensional Ising-like model. The energy of the given configuration $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ is equal to $\mid lnl_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} \mid$. The partition function $Z(\beta)$ is defined [12–15] as

$$Z_n(\beta) = \sum_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} l^{\beta}_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} = \sum_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} e^{-\beta|lnl_{\epsilon_1,\epsilon_2,\ldots\epsilon_n}|}, \tag{10}$$

where $\beta \in (-\infty, \infty)$ is a free parameter - the inverse "temperature". In the limit $n \to \infty$ the sum behaves as

$$Z(\beta) = e^{-n\beta F(\beta)}, \tag{11}$$

which defines the free energy, $F(\beta)$. The partition function $Z(\beta)$ can be alternatively written

$$Z(\beta) = \int d\lambda e^{nS(\lambda)-n\lambda\beta} \tag{12}$$

The entropy $S(\lambda)$ is the Legandre transform

$$S(\lambda) = -\beta F(\beta) + \lambda\beta, \tag{13}$$

where the relation between λ and β is obtained from

$$\lambda = \frac{d}{d\beta}(\beta F(\beta)), \quad \beta(\lambda) = S'(\lambda), \tag{14}$$

that have the following meaning: In the limit $n \to \infty$, $e^{nS(\lambda)}$ is the number of cylinders with length $l = e^{-n\lambda}$ or, equivalently, with local Lyapunov exponent λ. The Hausdorf dimension of the set of points in I having local Lyapunov exponent λ is $S(\lambda)/\lambda$.

We point out that the above defined one-dimensional Ising-like model has no direct physical meaning and is used here for the computation of the multifractal spectrum $S(\lambda)$ of the local Lyapunov exponents λ of the map (Eqs.(6), (9)).

By using Eqs.(6), (9), (10), (11) we numerically calculate the free energy at the point $K = -0.5, Q = 0.8, \gamma = 3$ (Fig.3). One can see from Fig.3 that the free energy has a nonanalytic behavior around $\beta_c \approx -1$, which shows the existence of the first order phase transition in this regions of β.

Large deviations of fluctuations of local Lyapunov exponents can be described by means of $S(\lambda)$. To consider the above results in terms of the entropy function $S(\lambda)$, let us first discuss the general view of the entropy function. First of all, it should be positive on some interval $[\lambda_{min}, \lambda_{max}]$. The value $\lambda = ln2$ must belong to that interval, which follows from the fact that the sum of the lengths of all cylinders on a given level is 1. Secondly it is often found that the values of λ_{min} and λ_{max} are given by the logarithms of the slopes at the origin .

The precise form of the entropy function is not easy to obtain with great accuracy. The existence of the first order phase transition implies that there should be a straight line segment in $S(\lambda)$ and the slope of the line equal to β_c. This scenario is seen in Fig. 4. The curve in the figure corresponds to $n = 13$. Of course, with the finite-size data, it is impossible to determine the straight line segment in $S(\lambda)$ and the straight line will increase with increasing n.

5. CONCLUSION

In this paper we have investigated the Q-state Potts model on the Bethe lattice in an external magnetic field. A strong connection with results from the theory of dynamical systems including chaos has been pointed out for $Q < 2$.

The local Lyapunov exponents are introduced as order parameters for characterizing the large variety of phase transitions which take place in Potts model. For certain values of parameters the distribution of local Lyapunov exponents are obtained by using the thermo-

dynamic formalism of multifractal. The scaling in the distribution of the local Lyapunov exponents are interpreted as a phase transition in the thermodynamics of the one dimensional Ising-like model. This phase transition is analyzed in terms of the "temperature" and local Lyapunov exponents λ.

We remark that similar behavior have been found in three-site interacting Ising model in the Husimi three [16,17].

The non-integer ($Q < 1$) valued Potts model is connected to the gelatation and vulcanization of branched polymers [18]. A dense Mandelbrot set of Fisher's zeroes for the non-integer valued Potts model can be obtained as the three-site antiferromagnetic interaction Ising model [19]. Few monolayers of polymers are described by higher-dimensional maps [20].The thin films of branched polymers can be regarded as the critical behavior of period p-tuplings in the coupled 1D maps [21]. The investigation of modulated phases and chaotic properties in polymers would be discussed in further publications.

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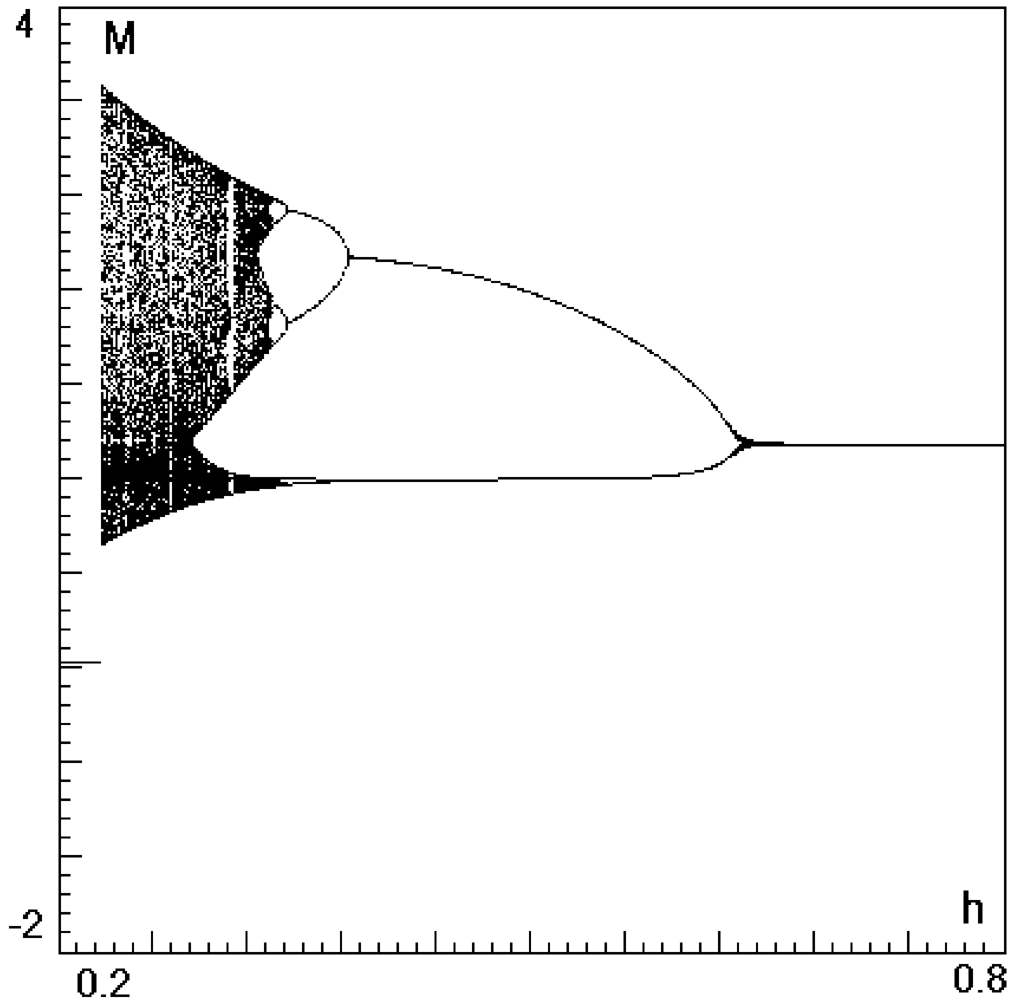

FIG. 1. Plot of m - magnetization versus h - external magnetic field ($K = -0.5$,$Q = 0.8$, $\gamma = 3$).

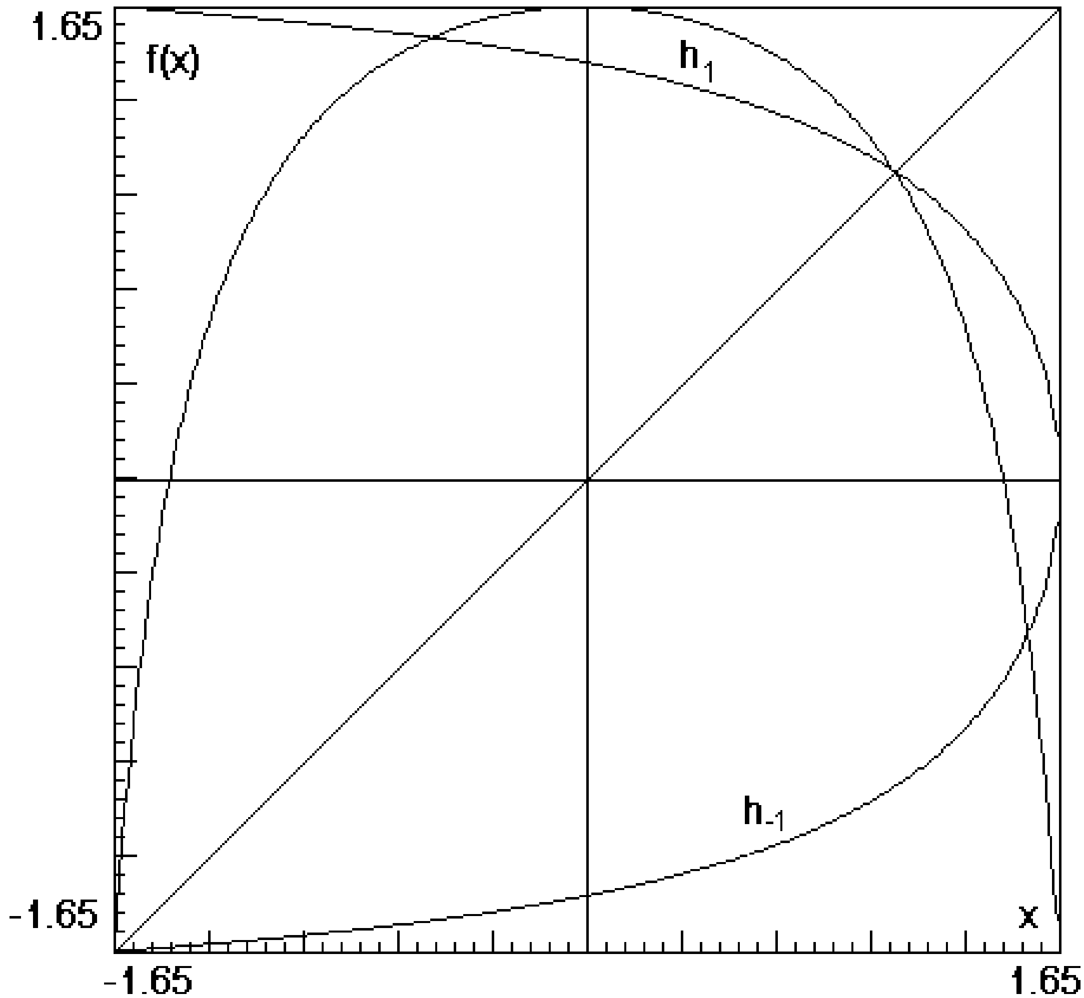

FIG. 2. The function of Eq.(6) for $K = -0.5$, $Q = 0.8$, $h = 0.23$ ($\gamma = 3$).

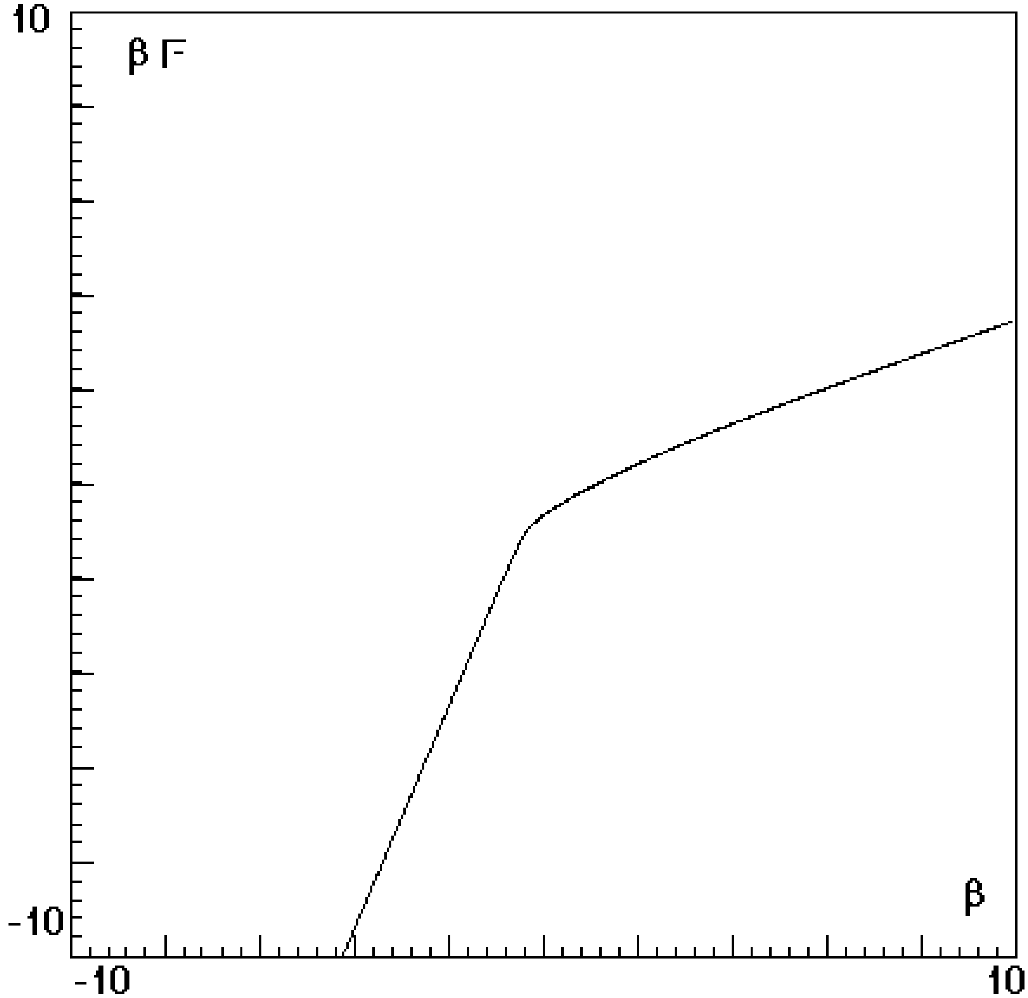

FIG. 3. $F(\beta)$, for $K = -0.5$, $Q = 0.8$, $h = 0.23$ ($\gamma = 3$).

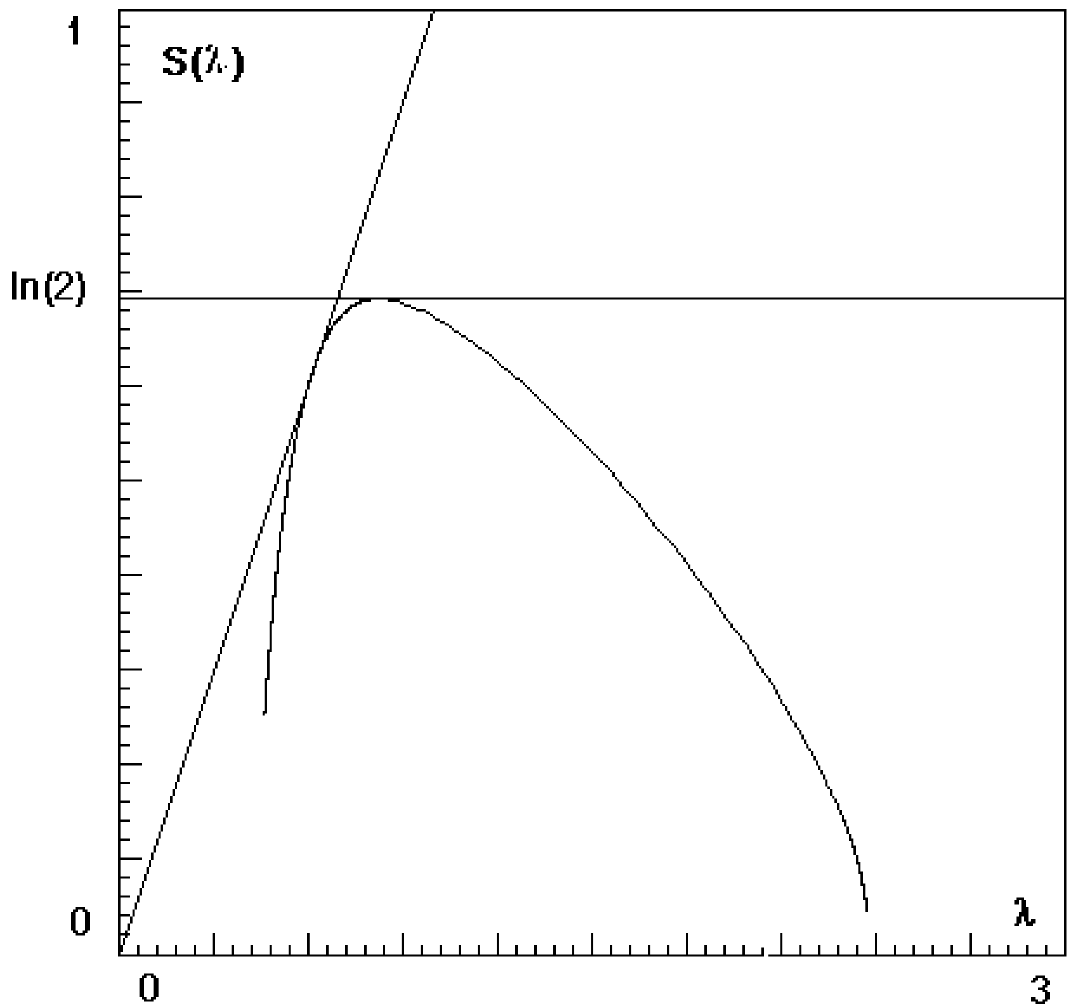

FIG. 4. $S(\lambda)$ corresponding to $n = 13$ for $K = -0.5$, $Q = 0.8$, $h = 0.23$ ($\gamma = 3$).